Exercises N7 01.04.2025 Electromechanics - Solutions

7.2 The material is mechanically free in all directions except for X_3 therefore, stress tensor has the following form:

$$\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0,$$
$$\sigma_3 = -p.$$

The linear contraction along x_3 direction is $\frac{\Delta L}{L} = -\varepsilon_{33}$, and Young modulus can be found as:

$$Y = \frac{p}{\Delta L/L} = \frac{\sigma_3}{\varepsilon_3}.$$

Constitutive equations are:

$$D_i = \varepsilon_0 K_{ij} E_j + d_{ij}\sigma_j,$$
$$\varepsilon_i = d_{ji} E_j + s_{ij}\sigma_j.$$

When, among the stress components, only $\sigma_3 \neq 0$, the equation for ε_3 is rewritten as

$$\varepsilon_3 = d_{j3}E_j + s_{3j}\sigma_j = d_{13}E_1 + d_{23}E_2 + d_{33}E_3 + s_{33}\sigma_3$$

Elements of symmetry of the material: mirror plane (001), mirror plane (010), and 2-fold axis [100]. Thus, the material has $mm2$ symmetry.

Note that the 2-fold axis is directed along [100]!

The piezoelectric tensor for $mm2$ symmetry, for the reference frame where the 2-fold axis is parallel to [001] direction, has the following form (see Symmetry Tables):

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \end{pmatrix}.$$

This tensor is not for the reference frame of the problem!

To make the 2-fold axis be directed along [100], one should make e.g. following transformation of the reference frame:

$x_1 \rightarrow -x_3$, $x_2 \rightarrow x_2$, $x_3 \rightarrow x_1$ (rotation by $90°$ with respect to $[010]$)

Then, the piezoelectric tensor transforms into:

$$d = \begin{pmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{26} \\ 0 & 0 & 0 & 0 & d_{35} & 0 \end{pmatrix}.$$

With these symmetry restrictions, $d_{23} = d_{33} = 0$, and the constitutive equation for ε_3 is rewritten as follows:

$$\varepsilon_3 = d_{13}E_1 + s_{33}\sigma_3$$

In order to obtain relationship $Y = \sigma_3/\varepsilon_3$, one has to know the value of the electric field E_1.

I. In sample I, where the (100) surfaces are not electroded and not connected, the induction D_1 must be zero since the surfaces cannot exchange their charges. The electric displacement D_1 is given by the equation:

$$D_1 = \varepsilon_0 K_{11} E_1 + \varepsilon_0 K_{12} E_2 + \varepsilon_0 K_{13} E_3 + d_{1n}\sigma_n$$

$d_{1n}\sigma_n = d_{13}\sigma_3$ since, among stress components, only $\sigma_3 \neq 0$. In the coordinate system of the problem, the symmetry restrictions of *mm2* group impose $K_{12} = K_{13} = 0$. Therefore,

$$D_1 = \varepsilon_0 K_{11} E_1 + d_{13}\sigma_3 = 0 \quad \Rightarrow \quad E_1 = -\frac{d_{13}}{\varepsilon_0 K_{11}}\sigma_3,$$

$$\varepsilon_3 = d_{13} E_1 + s_{33}\sigma_3 = \left(s_{33} - \frac{d_{13}^2}{\varepsilon_0 K_{11}}\right)\sigma_3,$$

$$Y = \frac{1}{s_{33} - \frac{d_{13}^2}{\varepsilon_0 K_{11}}}.$$

II. In sample II, the (100) surfaces are electroded and connected, so $E_1 = 0$.

$$\varepsilon_3 = s_{33}\sigma_3 \quad \Rightarrow \quad Y = \frac{1}{s_{33}}.$$

Since $\frac{d_{13}^2}{\varepsilon_0 K_{11}} > 0$, one can conclude that in sample I the measured Young modulus is larger.